Exercise 7

In Exercises 6 through 11, use the formal method, involving an infinite series of residues and illustrated in Examples 2 and 3 in Sec. 89, to find the function $f(t)$ that corresponds to the given function $F(s)$.

$$F(s) = \frac{1}{s\cosh s^{1/2}}.$$

$$\textit{Ans. } f(t) = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\exp\left[-\frac{(2n-1)^2\pi^2 t}{4}\right].$$

Solution

The inverse Laplace transform of the given function for $F(s)$ is defined by the Bromwich integral,

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\frac{ds}{s\cosh s^{1/2}},$$

where γ is a real constant chosen such that all singularities of the integrand lie to the left of the infinite vertical line $(\gamma - i\infty, \gamma + i\infty)$ in the complex plane. They occur where the denominator is equal to zero.

$$\begin{aligned} s\cosh s^{1/2} &= 0 \\ s = 0 \quad \text{or} \quad \cosh s^{1/2} &= 0 \\ \cos is^{1/2} &= 0 \\ is^{1/2} &= \frac{1}{2}(2k-1)\pi, \quad k = 0, \pm 1, \pm 2, \dots \quad \rightarrow \quad s_n = -\frac{\pi^2}{4}(2n-1)^2, \quad n = 1, 2, \dots \end{aligned}$$

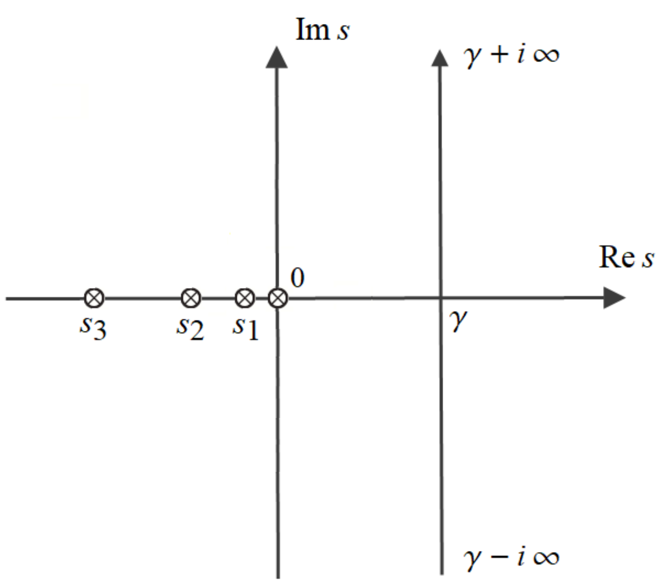

Figure 1: This is the complex plane with the singularities of the integrand marked as well as the vertical line $(\gamma - i\infty, \gamma + i\infty)$.

The integral is evaluated by considering a closed loop integral in the complex plane containing this vertical line and then applying the Cauchy residue theorem to get an equation, allowing us to solve for it. Normally the vertical line loops around back to $\gamma - iR$ by a semicircular arc to the left, but because of $s^{1/2} = \sqrt{s}$, a different path has to be taken. This is because for complex s, the square root function can be written in terms of the logarithm.

$$\sqrt{s} = \exp\left(\frac{1}{2}\log s\right)$$

In order for $\cosh s^{1/2}$ to be defined when s is negative, we will choose the following arbitrary branch cut, not the principal one ($|s| > 0,\ -\pi < \operatorname{Arg} s < \pi$).

$$\begin{aligned} &= \exp\left[\frac{1}{2}(\ln r + i\theta)\right], \quad \left(|s| > 0,\ -\frac{7\pi}{6} < \theta < \frac{5\pi}{6}\right) \\ &= \sqrt{r}e^{i\theta/2}, \end{aligned}$$

where $r = |s|$. Taking the branch cut into account, the closed loop in Figure 2 will be considered.

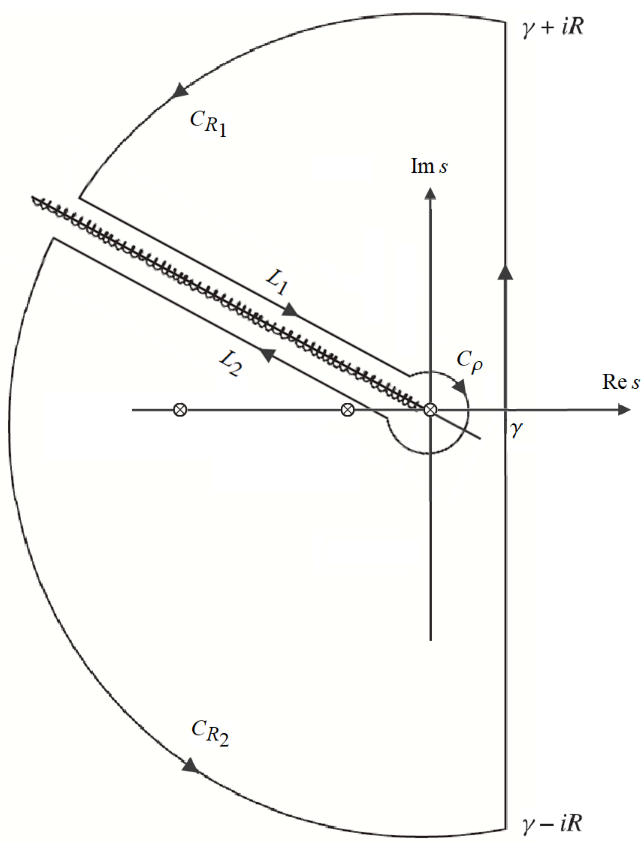

Figure 2: This is the closed loop that will be considered to calculate the inverse Laplace transform. The branch cut ($|s| > 0,\ -7\pi/6 < \arg\ s < 5\pi/6$) is represented in the complex plane by the squiggly line. In order to close the integration path after traversing the vertical line, let it follow a circular arc C_{R_1}. Once the path gets to the branch cut at $\theta = 5\pi/6$, integrate around it by going radially along L_1, around the origin by a circular arc C_ρ to the underside of the cut, and then radially again along L_2 at $\theta = -7\pi/6$. From there, let it follow a circular path C_{R_2} back to $\gamma - iR$.

Now that the integration path is closed, the Cauchy residue theorem can be applied, which states that the integral over this path is equal to $2\pi i$ times the sum of the residues inside the loop.

$$\oint_C e^{st}\frac{ds}{s\cosh s^{1/2}} = 2\pi i\sum_n \operatorname*{Res}_{s=s_n} e^{st}\frac{1}{s\cosh s^{1/2}}$$

This closed loop integral is the sum of six integrals, one over each arc in the loop.

$$\begin{aligned}\int_{\gamma-iR}^{\gamma+iR} e^{st}\frac{ds}{s\cosh s^{1/2}} + \int_{C_{R_1}} e^{st}\frac{ds}{s\cosh s^{1/2}} + \int_{L_1} e^{st}\frac{ds}{s\cosh s^{1/2}}\\ + \int_{C_\rho} e^{st}\frac{ds}{s\cosh s^{1/2}} + \int_{L_2} e^{st}\frac{ds}{s\cosh s^{1/2}} + \int_{C_{R_2}} e^{st}\frac{ds}{s\cosh s^{1/2}} = 2\pi i\sum_n \operatorname*{Res}_{s=s_n} e^{st}\frac{1}{s\cosh s^{1/2}}\end{aligned}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} C_{R_1}:&\quad s = Re^{i\theta}, && \theta = \frac{\pi}{2} \quad\to\quad \theta = \frac{5\pi}{6}\\ C_{R_2}:&\quad s = Re^{i\theta}, && \theta = -\frac{7\pi}{6} \quad\to\quad \theta = -\frac{\pi}{2}\\ C_\rho:&\quad s = \rho e^{i\theta}, && \theta = \frac{5\pi}{6} \quad\to\quad \theta = -\frac{7\pi}{6}\\ L_1:&\quad s = re^{5i\pi/6}, && r = R \quad\to\quad r = \rho\\ L_2:&\quad s = re^{-7i\pi/6}, && r = \rho \quad\to\quad r = R \end{aligned}$$

In the limit as $R\to\infty$ the integrals over C_{R_1} and C_{R_2} vanish, and in the limit as $\rho\to 0$ the integral over C_ρ tends to $-2\pi i$. The integral over L_1 cancels with the one over L_2. Proof for each of these statements will be given at the end.

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{ds}{s\cosh s^{1/2}} - 2\pi i = 2\pi i\sum_{n=1}^{\infty} \operatorname*{Res}_{s=s_n} e^{st}\frac{1}{s\cosh s^{1/2}}$$

The residues at $s = s_n$ are evaluated by

$$\operatorname*{Res}_{s=s_n} e^{st}\frac{1}{s\cosh s^{1/2}} = \frac{p(s_n)}{q'(s_n)},$$

where

$$\begin{aligned} p(s) &= e^{st}\\ q(s) &= s\cosh s^{1/2} \quad\to\quad q'(s) = \cosh s^{1/2} + s\sinh s^{1/2}\cdot\frac{1}{2}s^{-1/2}\\ &\qquad\qquad\qquad\qquad\quad = \cosh s^{1/2} + \frac{1}{2}s^{1/2}\sinh s^{1/2}. \end{aligned}$$

The $\cosh s^{1/2}$ term vanishes upon substituting s_n. We have

$$\begin{aligned}\frac{p(s_n)}{q'(s_n)} &= e^{s_n t}\frac{1}{\frac{1}{2}s_n^{1/2}\sinh s_n^{1/2}} \\ &= \exp\left[-\frac{\pi^2}{4}(2n-1)^2 t\right]\frac{1}{\frac{1}{2}\cdot\frac{1}{2i}(2n-1)\pi\sinh\left[\frac{1}{2i}(2n-1)\pi\right]} \\ &= \exp\left[-\frac{\pi^2}{4}(2n-1)^2 t\right]\frac{1}{\frac{1}{2}\cdot\frac{i}{2}(2n-1)\pi\sinh\left[\frac{i}{2}(2n-1)\pi\right]} \\ &= \exp\left[-\frac{\pi^2}{4}(2n-1)^2 t\right]\frac{1}{\frac{1}{2}\cdot\frac{i^2}{2}(2n-1)\pi\sin\left[\frac{1}{2}(2n-1)\pi\right]}.\end{aligned}$$

$(-1)^{n-1}$ can be used in place of $\sin\left[\frac{1}{2}(2n-1)\pi\right]$. Combining it with the minus sign from i^2 gives $(-1)^n$.

$$= \exp\left[-\frac{\pi^2}{4}(2n-1)^2 t\right]\frac{4(-1)^n}{(2n-1)\pi}$$

Consequently,

$$\operatorname*{Res}_{s=s_n} e^{st}\frac{1}{s\cosh s^{1/2}} = \frac{4(-1)^n}{\pi(2n-1)}\exp\left[-\frac{\pi^2}{4}(2n-1)^2 t\right]$$

and

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{ds}{s\cosh s^{1/2}} - 2\pi i = 2\pi i\sum_{n=1}^{\infty}\frac{4(-1)^n}{\pi(2n-1)}\exp\left[-\frac{\pi^2}{4}(2n-1)^2 t\right].$$

Divide both sides by $2\pi i$.

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{ds}{s\cosh s^{1/2}} - 1 = \sum_{n=1}^{\infty}\frac{4(-1)^n}{\pi(2n-1)}\exp\left[-\frac{\pi^2}{4}(2n-1)^2 t\right]$$

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{ds}{s\cosh s^{1/2}} = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\exp\left[-\frac{\pi^2}{4}(2n-1)^2 t\right]$$

Therefore, the inverse Laplace transform of $F(s)$ is

$$\boxed{f(t) = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\exp\left[-\frac{\pi^2}{4}(2n-1)^2 t\right].}$$

The Integral Over C_{R_1}

The objective here is to show that

$$\lim_{R\to\infty}\int_{C_{R_1}} e^{st}\frac{ds}{s\cosh s^{1/2}} = 0.$$

Start by writing hyperbolic cosine in terms of exponential functions.

$$\begin{aligned}\int_{C_{R_1}} e^{st}\frac{ds}{s\cosh s^{1/2}} &= \int_{C_{R_1}} e^{st}\frac{ds}{s\frac{\exp(s^{1/2})+\exp(-s^{1/2})}{2}}\\ &= \int_{C_{R_1}} e^{st}\frac{2\,ds}{s[\exp(s^{1/2})+\exp(-s^{1/2})]}\end{aligned}$$

The parameterization of the circular arc C_{R_1} in Figure 2 is $s = Re^{i\theta}$, where θ goes from $\pi/2$ to $5\pi/6$.

$$\begin{aligned}&= \int_{\pi/2}^{5\pi/6} e^{Re^{i\theta}t}\frac{2(Rie^{i\theta}\,d\theta)}{Re^{i\theta}[\exp(\sqrt{R}e^{i\theta/2})+\exp(-\sqrt{R}e^{i\theta/2})]}\\ &= \int_{\pi/2}^{5\pi/6} e^{Rt(\cos\theta+i\sin\theta)}\frac{2i\,d\theta}{\exp\left[\sqrt{R}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right]+\exp\left[-\sqrt{R}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right]}\\ &= \int_{\pi/2}^{5\pi/6} e^{Rt\cos\theta}e^{iRt\sin\theta}\frac{2i\,d\theta}{\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(i\sqrt{R}\sin\frac{\theta}{2}\right)+\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(-i\sqrt{R}\sin\frac{\theta}{2}\right)}\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_{R_1}} e^{st}\frac{ds}{s\cosh s^{1/2}}\right| &= \left|\int_{\pi/2}^{5\pi/6} e^{Rt\cos\theta}e^{iRt\sin\theta}\frac{2i\,d\theta}{\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(i\sqrt{R}\sin\frac{\theta}{2}\right)+\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(-i\sqrt{R}\sin\frac{\theta}{2}\right)}\right|\\ &\le \int_{\pi/2}^{5\pi/6}\left|e^{Rt\cos\theta}e^{iRt\sin\theta}\frac{2i}{\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(i\sqrt{R}\sin\frac{\theta}{2}\right)+\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(-i\sqrt{R}\sin\frac{\theta}{2}\right)}\right|d\theta\\ &= \int_{\pi/2}^{5\pi/6}\left|e^{Rt\cos\theta}\right|\left|e^{iRt\sin\theta}\right|\frac{|2i|}{\left|\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(i\sqrt{R}\sin\frac{\theta}{2}\right)+\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(-i\sqrt{R}\sin\frac{\theta}{2}\right)\right|}\,d\theta\end{aligned}$$

$$
\begin{aligned}
\left|\int_{C_{R_1}} e^{st}\frac{ds}{s\cosh s^{1/2}}\right| &\leq \int_{\pi/2}^{5\pi/6} e^{Rt\cos\theta}\frac{2}{\left|\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(i\sqrt{R}\sin\frac{\theta}{2}\right)+\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(-i\sqrt{R}\sin\frac{\theta}{2}\right)\right|}d\theta \\
&\leq \int_{\pi/2}^{5\pi/6} e^{Rt\cos\theta}\frac{2}{\left|\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(i\sqrt{R}\sin\frac{\theta}{2}\right)\right|-\left|\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(-i\sqrt{R}\sin\frac{\theta}{2}\right)\right|}\,d\theta \\
&= \int_{\pi/2}^{5\pi/6} e^{Rt\cos\theta}\frac{2}{\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)-\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)}\,d\theta
\end{aligned}
$$

Take the limit of both sides now as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_{R_1}} e^{st}\frac{ds}{s\cosh s^{1/2}}\right| \leq \lim_{R\to\infty}\int_{\pi/2}^{5\pi/6} e^{Rt\cos\theta}\frac{2}{\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)-\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)}\,d\theta$$

Because the limits of integration are constant, the limit may be brought inside the integral.

$$\lim_{R\to\infty}\left|\int_{C_{R_1}} e^{st}\frac{ds}{s\cosh s^{1/2}}\right| \leq \int_{\pi/2}^{5\pi/6} \lim_{R\to\infty} e^{Rt\cos\theta}\frac{2}{\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)-\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)}\,d\theta$$

Since θ is between $\pi/2$ and $5\pi/6$, the cosine of θ is negative and the cosine of $\theta/2$ is positive. In addition, R and t are positive, so $Rt\cos\theta$ tends to $-\infty$, $\sqrt{R}\cos(\theta/2)$ tends to ∞, and $-\sqrt{R}\cos(\theta/2)$ tends to $-\infty$. As a result, the integral tends to zero.

$$\lim_{R\to\infty}\left|\int_{C_{R_1}} e^{st}\frac{ds}{s\cosh s^{1/2}}\right| \leq 0$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty}\left|\int_{C_{R_1}} e^{st}\frac{ds}{s\cosh s^{1/2}}\right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_{R_1}} e^{st}\frac{ds}{s\cosh s^{1/2}} = 0.$$

The Integral Over C_{R_2}

The objective here is to show that

$$\lim_{R\to\infty}\int_{C_{R_2}} e^{st}\frac{ds}{s\cosh s^{1/2}} = 0.$$

Start by writing hyperbolic cosine in terms of exponential functions.

$$\begin{aligned}\int_{C_{R_2}} e^{st}\frac{ds}{s\cosh s^{1/2}} &= \int_{C_{R_2}} e^{st}\frac{ds}{s\frac{\exp(s^{1/2})+\exp(-s^{1/2})}{2}} \\ &= \int_{C_{R_2}} e^{st}\frac{2\,ds}{s[\exp(s^{1/2})+\exp(-s^{1/2})]}\end{aligned}$$

The parameterization of the circular arc C_{R_2} in Figure 2 is $s = Re^{i\theta}$, where θ goes from $-7\pi/6$ to $-\pi/2$.

$$\begin{aligned}&= \int_{-7\pi/6}^{-\pi/2} e^{Re^{i\theta}t}\frac{2(Rie^{i\theta}\,d\theta)}{Re^{i\theta}[\exp(\sqrt{R}e^{i\theta/2})+\exp(-\sqrt{R}e^{i\theta/2})]} \\ &= \int_{-7\pi/6}^{-\pi/2} e^{Rt(\cos\theta+i\sin\theta)}\frac{2i\,d\theta}{\exp\left[\sqrt{R}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right]+\exp\left[-\sqrt{R}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right]} \\ &= \int_{-7\pi/6}^{-\pi/2} e^{Rt\cos\theta}e^{iRt\sin\theta}\frac{2i\,d\theta}{\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(i\sqrt{R}\sin\frac{\theta}{2}\right)+\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(-i\sqrt{R}\sin\frac{\theta}{2}\right)}\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_{R_2}} e^{st}\frac{ds}{s\cosh s^{1/2}}\right| &= \left|\int_{-7\pi/6}^{-\pi/2} e^{Rt\cos\theta}e^{iRt\sin\theta}\frac{2i\,d\theta}{\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(i\sqrt{R}\sin\frac{\theta}{2}\right)+\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(-i\sqrt{R}\sin\frac{\theta}{2}\right)}\right| \\ &\le \int_{-7\pi/6}^{-\pi/2}\left|e^{Rt\cos\theta}e^{iRt\sin\theta}\frac{2i}{\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(i\sqrt{R}\sin\frac{\theta}{2}\right)+\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(-i\sqrt{R}\sin\frac{\theta}{2}\right)}\right|d\theta \\ &= \int_{-7\pi/6}^{-\pi/2}\left|e^{Rt\cos\theta}\right|\left|e^{iRt\sin\theta}\right|\frac{|2i|}{\left|\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(i\sqrt{R}\sin\frac{\theta}{2}\right)+\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(-i\sqrt{R}\sin\frac{\theta}{2}\right)\right|}d\theta\end{aligned}$$

$$\begin{aligned}
\left|\int_{C_{R_2}} e^{st}\frac{ds}{s\cosh s^{1/2}}\right| &\leq \int_{-7\pi/6}^{-\pi/2} e^{Rt\cos\theta}\frac{2}{\left|\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(i\sqrt{R}\sin\frac{\theta}{2}\right)+\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(-i\sqrt{R}\sin\frac{\theta}{2}\right)\right|}\,d\theta \\
&\leq \int_{-7\pi/6}^{-\pi/2} e^{Rt\cos\theta}\frac{2}{\left|\left|\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(i\sqrt{R}\sin\frac{\theta}{2}\right)\right|-\left|\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)\exp\left(-i\sqrt{R}\sin\frac{\theta}{2}\right)\right|\right|}\,d\theta \\
&= \int_{-7\pi/6}^{-\pi/2} e^{Rt\cos\theta}\frac{2}{\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)-\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)}\,d\theta
\end{aligned}$$

Take the limit of both sides now as $R\to\infty$.

$$\lim_{R\to\infty}\left|\int_{C_{R_2}} e^{st}\frac{ds}{s\cosh s^{1/2}}\right| \leq \lim_{R\to\infty}\int_{-7\pi/6}^{-\pi/2} e^{Rt\cos\theta}\frac{2}{\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)-\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)}\,d\theta$$

Because the limits of integration are constant, the limit may be brought inside the integral.

$$\lim_{R\to\infty}\left|\int_{C_{R_2}} e^{st}\frac{ds}{s\cosh s^{1/2}}\right| \leq \int_{-7\pi/6}^{-\pi/2} \lim_{R\to\infty} e^{Rt\cos\theta}\frac{2}{\exp\left(\sqrt{R}\cos\frac{\theta}{2}\right)-\exp\left(-\sqrt{R}\cos\frac{\theta}{2}\right)}\,d\theta$$

Since θ is between $-7\pi/6$ and $-\pi/2$, the cosine of θ is negative and the cosine of $\theta/2$ is either positive or negative. In addition, R and t are positive, so $Rt\cos\theta$ tends to $-\infty$, $\sqrt{R}\cos(\theta/2)$ tends to $\pm\infty$, and $-\sqrt{R}\cos(\theta/2)$ tends to $\mp\infty$. As a result, the integral tends to zero.

$$\lim_{R\to\infty}\left|\int_{C_{R_2}} e^{st}\frac{ds}{s\cosh s^{1/2}}\right| \leq 0$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty}\left|\int_{C_{R_2}} e^{st}\frac{ds}{s\cosh s^{1/2}}\right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_{R_2}} e^{st}\frac{ds}{s\cosh s^{1/2}} = 0.$$

The Integral Over C_ρ

The objective here is to show that

$$\lim_{\rho \to 0} \int_{C_\rho} e^{st} \frac{ds}{s \cosh s^{1/2}} = -2\pi i.$$

Start by writing hyperbolic cosine in terms of exponential functions.

$$\begin{aligned}
\int_{C_\rho} e^{st} \frac{ds}{s \cosh s^{1/2}} &= \int_{C_\rho} e^{st} \frac{ds}{s \frac{\exp(s^{1/2}) + \exp(-s^{1/2})}{2}} \\
&= \int_{C_\rho} e^{st} \frac{2\, ds}{s[\exp(s^{1/2}) + \exp(-s^{1/2})]}
\end{aligned}$$

The parameterization of the circular arc C_ρ in Figure 2 is $s = \rho e^{i\theta}$, where θ goes from $5\pi/6$ to $-7\pi/6$.

$$\begin{aligned}
&= \int_{5\pi/6}^{-7\pi/6} e^{\rho e^{i\theta} t} \frac{2(\rho i e^{i\theta}\, d\theta)}{\rho e^{i\theta}[\exp(\sqrt{\rho} e^{i\theta/2}) + \exp(-\sqrt{\rho} e^{i\theta/2})]} \\
&= \int_{5\pi/6}^{-7\pi/6} e^{\rho e^{i\theta} t} \frac{2i\, d\theta}{\exp(\sqrt{\rho} e^{i\theta/2}) + \exp(-\sqrt{\rho} e^{i\theta/2})}
\end{aligned}$$

Now take the limit of both sides as $\rho \to 0$.

$$\lim_{\rho \to 0} \int_{C_\rho} e^{st} \frac{ds}{s \cosh s^{1/2}} = \lim_{\rho \to 0} \int_{5\pi/6}^{-7\pi/6} e^{\rho e^{i\theta} t} \frac{2i\, d\theta}{\exp(\sqrt{\rho} e^{i\theta/2}) + \exp(-\sqrt{\rho} e^{i\theta/2})}$$

Because the limits of integration are constant, the limit may be brought inside the integral.

$$\begin{aligned}
\lim_{\rho \to 0} \int_{C_\rho} e^{st} \frac{ds}{s \cosh s^{1/2}} &= \int_{5\pi/6}^{-7\pi/6} \lim_{\rho \to 0} e^{\rho e^{i\theta} t} \frac{2i\, d\theta}{\exp(\sqrt{\rho} e^{i\theta/2}) + \exp(-\sqrt{\rho} e^{i\theta/2})} \\
&= \int_{5\pi/6}^{-7\pi/6} e^0 \frac{2i\, d\theta}{\exp(0) + \exp(0)} \\
&= \int_{5\pi/6}^{-7\pi/6} i\, d\theta \\
&= i\left[-\frac{7\pi}{6} - \frac{5\pi}{6}\right]
\end{aligned}$$

Therefore,

$$\lim_{\rho \to 0} \int_{C_\rho} e^{st} \frac{ds}{s \cosh s^{1/2}} = -2\pi i.$$

The Integral Over L_1

The parameterization of the radial arc L_1 in Figure 2 is $s = re^{5i\pi/6}$, where r goes from R to ρ.

$$\begin{aligned}
\int_{L_1} e^{st}\frac{ds}{s\cosh s^{1/2}} &= \int_R^\rho e^{re^{5i\pi/6}t}\frac{dr\,e^{5i\pi/6}}{re^{5i\pi/6}\cosh(\sqrt{r}e^{5i\pi/12})} \\
&= \int_R^\rho e^{re^{5i\pi/6}t}\frac{dr}{r\cosh(\sqrt{r}e^{5i\pi/12})} \\
&= \int_R^\rho \exp\left[rt\left(-\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\right]\frac{dr}{r\cosh\left[\sqrt{r}\left(\frac{\sqrt{6}-\sqrt{2}}{4}+i\frac{\sqrt{6}+\sqrt{2}}{4}\right)\right]} \\
&= -\int_\rho^R \exp\left[rt\left(-\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\right]\frac{dr}{r\cosh\left[\sqrt{r}\left(\frac{\sqrt{6}-\sqrt{2}}{4}+i\frac{\sqrt{6}+\sqrt{2}}{4}\right)\right]}
\end{aligned}$$

The Integral Over L_2

The parameterization of the radial arc L_2 in Figure 2 is $s = re^{-7i\pi/6}$, where r goes from ρ to R.

$$\begin{aligned}
\int_{L_2} e^{st}\frac{ds}{s\cosh s^{1/2}} &= \int_\rho^R e^{re^{-7i\pi/6}t}\frac{dr\,e^{-7i\pi/6}}{re^{-7i\pi/6}\cosh(\sqrt{r}e^{-7i\pi/12})} \\
&= \int_\rho^R e^{re^{-7i\pi/6}t}\frac{dr}{r\cosh(\sqrt{r}e^{-7i\pi/12})} \\
&= \int_\rho^R \exp\left[rt\left(-\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\right]\frac{dr}{r\cosh\left[\sqrt{r}\left(\frac{-\sqrt{6}+\sqrt{2}}{4}-i\frac{\sqrt{6}+\sqrt{2}}{4}\right)\right]} \\
&= \int_\rho^R \exp\left[rt\left(-\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\right]\frac{dr}{r\cosh\left[\sqrt{r}\left(\frac{\sqrt{6}-\sqrt{2}}{4}+i\frac{\sqrt{6}+\sqrt{2}}{4}\right)\right]}
\end{aligned}$$

Therefore,

$$\int_{L_1} e^{st}\frac{ds}{s\cosh s^{1/2}} + \int_{L_2} e^{st}\frac{ds}{s\cosh s^{1/2}} = 0.$$